IB Linear Algebra – Example Sheet 3

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1. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

The second and third matrices commute; find a basis with respect to which they are both diagonal.

2. By considering the rank or minimal polynomial of a suitable matrix, find the eigenvalues of the $n \times n$ real matrix A with each diagonal entry equal to λ and all other entries 1. Hence write down the determinant of A.

3. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.

4. Prove the Generalised Eigenspace Decomposition Theorem (stated in lectures).

5. Let α be an endomorphism of a complex vector space V. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\mathrm{Ker}(\alpha - \lambda \mathrm{id}_V)$ and $\mathrm{Ker}(\alpha^2 - \lambda^2 \mathrm{id}_V)$ necessarily the same?

6. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^{1000}$.

7. Without appealing directly to the uniqueness of Jordan Normal Form show that none of the following real matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is the matrix

$$\begin{pmatrix} -2 & -2 & -1 \\ 3 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

similar to any of them? If so, which? Find a basis such that it is in Jordan Normal Form.

8. Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.

9. Let $J_n(\lambda) \in \mathcal{M}_{n\times n}(\mathbb{C})$ be the n-by-n Jordan block with diagonal entry $\lambda \in \mathbb{C}$. Find the Jordan normal form of $J_n(\lambda)^k$ for every positive integer k.

(You may find it helpful to consider the case $\lambda = 0$ separately.)

10. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Without appealing to Jordan Normal Form, show that there is a vector y such that $(y, \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y))$ is a basis for V. What is the matrix representation of α with respect to this basis? And the matrix representation of α^k, for an arbitrary positive integer k?

Show that if β is an endomorphism of V which commutes with α, then $\beta = p(\alpha)$ for some polynomial p. *[Hint: consider $\beta(y)$.]* What is the form of the matrix for β with respect to the above basis?

11. (a) Let A be an invertible square matrix. Describe the eigenvalues and the characteristic and minimal polynomials of A^{-1} in terms of those of A.

    (b) Prove that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan Normal Form a Jordan block $J_m(\lambda^{-1})$. Use this to find the Jordan Normal Form of A^{-1}, for an invertible square matrix A.

    (c) Prove that any square complex matrix is similar to its transpose.

12. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

13. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_n & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \cdots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}.$$

    Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

Bonus/Optional Questions

1. (Cayley–Hamilton theorem for a general field) Let $\mathbb{F}$ be any field, $A \in M_{n\times n}(\mathbb{F})$, and $B = tI - A$. By considering the equation $B \cdot \operatorname{adj}(B) = \det(B)I$, or otherwise, prove that $\chi_A(A) = 0$, where $\chi_A(t)$ is the characteristic polynomial of A.

2. Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal polynomials of A? If A is not diagonalisable, what are the possible characteristic polynomials and JNFs of A? *(The list is quite long!)*

3. Give an example of a real vector space V and a linear map $\alpha \in \mathcal{L}(V, V)$ such that every real number is an eigenvalue of α.